PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
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The Generation and Propagation of Ocean Waves and Swell. I. Wave Periods and Velocities

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THE GENERATION AND PROPAGATION OF OCEAN WAVES AND SWELL

I. WAVE PERIODS AND VELOCITIES

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CONTENTS

A method is described for measuring and examining ocean waves in a way which allows their amplitude and period to be determined with some precision. Data obtained in this way are compared with meteorological charts of the ocean in an attempt to assess the velocity of propagation of swell over long distances. A critical estimate is only possible when the meteorological conditions are sufficiently simple, but in one selected example it appears that the velocity of propagation is within 5 % of the value prescribed by hydrodynamical theory. The evidence in more complicated instances does not disagree with this result, but does not permit of such an exact interpretation.

The waves are measured by the fluctuating pressure which they produce upon an instrument laid on the sea bed in shallow water near the coast. The resulting curves are examined by a machine which draws the frequency spectrum of the recorded waves. The information given by these spectra is combined with the information of wind strength given by the meteorological charts to form a 'propagation diagram' whose appearance is a test of the validity of the theoretical group velocity. A suitable theoretical basis is given to the work.

Three examples are discussed in detail: a depression in the North Atlantic, a tropical storm off the coast of Florida and a train of swell which appears to have originated in a storm off Cape Horn. The swell in these examples had travelled to Land's End distances of 1200, 2800 and 6000 miles respectively.

The paper deals only with the velocity of propagation of the swell and does not discuss how the amplitude of the swell may depend upon the distance the swell has travelled or the wind strength and fetch in the generating area.

Introduction

The first authority to make systematic swell forecasts was the French swell-prediction service in French Morocco. With the help of meteorological charts, observers in the Azores and on the coast of Portugal, and reports from ships, they were able to trace any outstanding swell that reached the coast of Morocco to its origin—generally in the wind associated with a depression moving across the North Atlantic Ocean. Various types of depressions were classified, and with the help of rules which included some allowance for the effect of favourable or contrary winds over the ocean through which the swell had to travel, they were able to make useful predictions of the state of an anchorage. The foundation of the predictions remained primarily the reports from the Azores.

During the war the Naval Meteorological Branch had to develop a method of predicting waves and swell from meteorological charts and forecasts. The empirical formulae put forward during the past 100 years proved inadequate, but by making the best use of previous data, and filling the gaps as quickly as possible with new observations, useful tables were compiled, and the first instructions for predicting waves and swell were issued in 1942. Similar work was undertaken in the United States of America, and although detailed agreement was not reached the Admiralty Swell Forecasting Section, staffed with United States and British officers, was able to make reasonably accurate predictions.

The starting-point of a prediction was an isobaric chart in which *generating areas*, where the wind was blowing towards the scene of the prediction or within 30° of the direct line, were distinguished. The average wind speed was estimated from the isobar spacing, some allowance being made for the lesser speeds at the beginning and end of the storm. Account was also taken of the *fetch*, or length of the generating area, and the *duration*, which included some allowance for the waves already present when the wind started to blow. For an exposed area allowance had to be made for ocean swell from very distant storm areas, and for the effect of favourable and contrary winds over the intervening ocean. If the prediction was to include an estimate of breakers and surf some general information was required of the depth contours and beach slope.

The experience gained in this work emphasized the need for further measurement and theoretical study of waves and swell, and long-term research was started in Great Britain and the United States. In this country the first wave-recorders were developed in the Mine Design Department, and routine wave recordings were made at a number of stations on the south and west coast in the months preceding the invasions of Normandy. They were used mainly for comparison with predictions from meteorological charts, and for measurements in connexion with amphibious exercises and the development of artificial breakwaters.

In June 1944 the responsibility for the measurement and theory of waves was placed with a newly formed oceanographical section at the Admiralty Research Laboratory, where this report has been prepared. It deals mainly with basic methods and information that are necessary before a wave or swell component in a record obtained in a coastal region can be traced with confidence to its origin in a local or distant storm area.

Theory of waves and swell

In drawing up a theoretical outline of the behaviour of swell it is desirable to idealize very greatly the actual conditions that obtain in the ocean. A storm of wide extent and long duration produces effects which it is perhaps possible to look upon as being the sum of the effects of a large number of atmospheric disturbances of short duration and of small area. Two theoretical examples of waves spreading from a localized disturbance on the surface of a wide expanse of fluid are due to Cauchy (1827) and Poisson (1816), one treating the disturbance as an instantaneous pressure point and the other treating the disturbance as an instantaneous surface elevation. The two examples differ in the amplitude they ascribe to the resulting waves, but both examples show that the surface elevation η of the fluid at time t after the disturbance, and at distance R from the centre of disturbance, is of the form

$$\eta = \frac{1}{R} f\left(\frac{t}{R}\right) \sin\left(\frac{gt^2}{4R}+\epsilon\right)$$

$$= \frac{1}{R} f\left(\frac{t}{R}\right) \sin\alpha(t, R),$$

where $f(t/R)$ is a slowly varying function and ϵ is a phase angle.

At some distance from the centre of disturbance this expression represents a train of waves travelling outwards from the disturbance, whose amplitude wave-length and period are given by

$$\text{amplitude} = \frac{1}{R} f\left(\frac{t}{R}\right),$$

$$\text{wave-length} = -2\pi \Big/ \frac{d\alpha}{dR} = 8\pi R^2/gt^2,$$

$$\text{period} = 2\pi \Big/ \frac{d\alpha}{dt} = 4\pi R/gt.$$

At a given instant the waves have a greater wave-length and period the greater their distance from the centre of disturbance, and at any given place the length and period of the waves will decrease with the passage of time. This is in agreement with the theory of regular wave trains which shows that the 'group velocity' of waves increases with the wave period. One of the more important implications of the theories of Cauchy and Poisson is that waves of a certain period arrive at a point on the fluid as if they had travelled from the centre of the disturbance with the group velocity appropriate to their period. It can be shown from Kelvin's principle of stationary phase that this applies to both deep and shallow water, and an interesting numerical example is given by Green (1909) and quoted by Havelock (1914). In deep water the group velocity is half the wave velocity and is proportional to the wave period by the relation

$$\text{group velocity} = \tfrac{1}{2}\text{ wave velocity} = \frac{g}{4\pi} \times \text{wave period}.$$

This theory of wave propagation from a small disturbance only holds when the waves are of small height and the sea behaves as an ideal fluid. Under these conditions the motion of the fluid due to a number of disturbances is obtained by mere addition, and it would be possible to regard the effect of a wide disturbance as the sum of the effects of a number of localized disturbances. It will be observed that the waves arriving from a single localized disturbance have a single period at any one time and place. The waves from a widespread disturbance may be expected to exhibit a more or less narrow range of periods.

In practice the measurement of the period of swell is often confused by the presence of different trains of swell arriving from different storms or by the presence of wind waves generated locally. Some method of identifying and measuring these different periods is necessary. This may be done by making a frequency analysis of the recorded wave motion.

(a) *The wave propagation diagram*

A convenient way of dealing with a series of observations of wave period is to construct a diagram of the form shown in figure 1. The ordinate t represents time and the abscissa the distance from the measuring station. When waves of period T_1 are observed at time t_1 their previous history is represented on the diagram by a straight line drawn from the point $(0, t_1)$ at an inclination to the time axis corresponding to the group velocity of waves of period T_1. Observations at later times give further lines whose convergence marks the time and place of the disturbance from which the waves originate. In practice the swell is generated in a storm of finite area and duration, and the observed swell exhibits not a single period but a band of periods. The method adopted has been that of drawing lines appropriate to the maximum and minimum period measured at each time, and it is found that the two systems of lines envelope the storm area in a manner which will be discussed in greater detail in the consideration of examples of propagation.

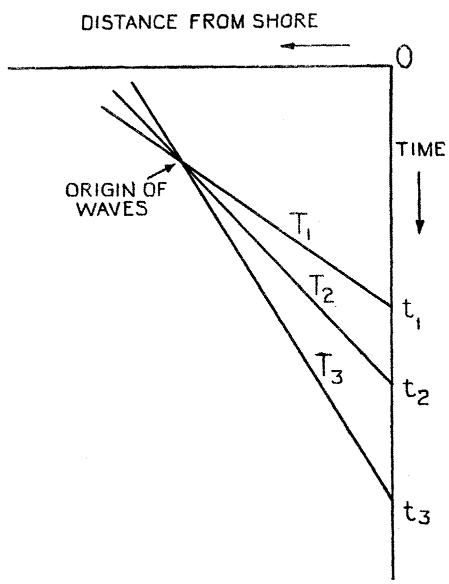

FIGURE 1. Wave propagation diagram.

(b) *The amplitude of the swell or waves*

The theories of Cauchy and Poisson lead to the conclusion that the amplitude of a group of waves travelling away from a localized disturbance should vary inversely as the distance they have travelled. The effects of viscosity, breaking in cross seas, and of opposing winds may modify this result for waves that radiate from a storm. The theory does not give any indication of the distribution of energy to be expected in practice in the wave spectrum nor any indication of how the wave energy at any place may be expected to vary with time. These depend upon the way in which the waves are generated and must be determined by observation. In the present article it has been thought best to work only on some selected examples and to attempt to get critical evidence for the rate of travel of swell. A knowledge of this rate of travel is essential in correlating the observed amplitude of swell with the wind strength fetch and duration of the storm.

(c) *Possible deviations from classical theory*

Ocean waves and swell do not necessarily behave in the manner prescribed by theory. The application of classical theory to the sea involves the following assumptions, none of which is fully justified:

(1) The water behaves as a uniform, non-viscous, incompressible fluid.

(2) The swell and waves are always of low height and combine linearly.

(3) The resistance of the air is negligible.

(4) The rotation of the earth has no effect on the propagation of the waves.

These assumptions are discussed below. It appears likely that they hold sufficiently well for the classical theory to be used as a basis for discussing the propagation of swell from a distant storm.

(1) The effect of viscosity is treated by Lamb (1932, article 349), and it does not appear to affect the theoretical group velocity of the waves. Turbulence can be represented as an artificial 'eddy viscosity' so long as the size of the turbulent eddies is small compared with the length of the waves. Large-scale turbulence cannot be treated in this way. It may take the form of a horizontal circulation of the water which may scatter the wave train to some extent and lead to an overall reduction in the rate of travel of a wave group through the area. The turbulence may take the form of a general shearing motion in horizontal planes, and it is interesting to note that the theoretical waves described by Gerstner (1802) involve a small shearing motion but have nearly the same group velocity as the theoretical waves described by Stokes (1847) which are irrotational. Intermediate forms of shearing waves have been discussed by Miche (1944). It is possible to show that the change in group velocity produced by horizontal shearing is comparable to or less than the shearing velocity. Large-scale uniform motion of the water also affects the group velocity by an amount comparable with the uniform water velocity. All these forms of large-scale turbulence may be produced by ocean currents, but it is only in special areas of deep water, or in shallow water near the coast that the velocities are sufficiently large to produce a measurable effect on the velocity of propagation of swell.

(2) The theory does not apply to waves of finite height. Regular trains of waves of finite height have been studied by Gerstner (1802), Stokes (1847), Levi Civita (1925) and Struik (1926), but no solution has been given for the combination of such wave trains or for the manner of travel of groups of waves of finite height. The theory shows that the finite height of the waves in a regular train may produce a small increase in the wave speed, and it is possible that groups of high waves may be found to travel at a velocity slightly different from that of low waves of the same period. Any such difference would be more marked in swell during the early stages of its travel from the storm area; over the greater part of its journey the swell may be expected to behave as a train of waves of low height.

(3) When waves move through still air or against contrary winds the air moves relatively to the wave profile, and it can be expected that the surface pressure will vary between the crests and troughs, being least at the crests. This will lead to a decrease in wave velocity, but the velocity differences should be of the order of $\frac{1}{800}$, the ratio of the densities of the two fluids. This is too small to be measurable.

(4) The propagation of waves on a rotating earth is studied by Bjerknes, Bjerknes, Solberg & Bergeron (1933). They show that the velocity of waves of the period considered in this

paper is unaffected by the rotation of the earth. The possibility remains that the waves travel by a path larger than the great circle path between the generating area and the recording station, but the difference is likely to be small. If waves in travelling 6000 miles deviate by 100 miles from the great circle path the travel time will be lengthened by only half an hour.

These tentative assumptions are implicit in the present discussion of experimental results. They are least likely to hold good in the storm area where the waves are generated and more likely to be valid for the propagation of swell through calm areas of sea. In future work it may be possible to make some allowance for the deviation from simple theory.

(*d*) *Practical expectations*

In so far as they deal with the propagation of waves the conclusions from hydrodynamical theory can be summarized as follows.

Waves from different parts of a storm, or from different storms, should travel independently after they leave the storm area, with group velocities appropriate to their periods, and each storm should contribute its own frequency band to the measurements at any point of observation in the path of the swell. Since the group velocities are proportional to the periods so that the longer waves arrive at a distant observation point ahead of the shorter waves, subsequent observations should reveal a slow decrease in the mean period of the band, the decrease being a simple function of the time and position of the storm. The behaviour of long swell should conform most closely to these rules, and the agreement should be the least for waves raised by a local wind.

(*e*) *Observations*

If an attempt is to be made to demonstrate the existence of separate bands of periods and to compare their behaviour with the theoretical expectations, then because the character of the storms and swell may often be too complex to permit of a clear interpretation, it is very desirable that the information both of swell and of weather should be regular, frequent and reliable. An irregular series of infrequent observations is likely to be of little use.

Since the wave observations are to be subjected to harmonic analysis it is desirable to measure some quantity which is not only periodic in a simple wave train but also varies in a simple sinusoidal manner with this period. The displacement of a water particle is almost exactly simple harmonic in a train of regular waves even when these are of finite height, and might conceivably be measured by the motion of a freely floating buoy, but the existing instruments that are suitable for routine wave recording measure either the fluctuating pressure at a point on the sea bed or the fluctuating height of the water surface above an instrument on the sea bed. In a train of waves of finite height these quantities vary in a manner which is trochoidal rather than sinusoidal. The frequency analysis of the fluctuating pressure due to a regular train of steep waves would show not only the fundamental period but would also show small components with a period one-half and one-third of the fundamental period, and the experimenter might misinterpret this to mean the existence of wave trains of these periods. The corresponding errors would be larger if the record was of the elevation of the water surface. This matter is discussed in appendix 1, where it is shown that in the series of observations quoted later in the article the errors of this nature are unimportant.

There is some advantage in studying swell by means of a pressure recorder in deep water, for the instrument is relatively insensitive to the presence of wind waves of short wave-length. The pressure amplitude P due to a wave of surface amplitude is theoretically given by the relation

$$P = a\,\text{sech}\,2\pi h/\lambda,$$

where λ is the wave-length and h is the depth of water, and the pressure P is measured in equivalent head of water. The pressure amplitude is only a small fraction of the surface amplitude when the ratio h/λ is large, but is equal to the surface amplitude, when h/λ is small. A corollary is that due allowance for this factor must be made in estimating the surface amplitude of the swell from its observed pressure amplitude, and that due allowance will have to be made for the tidal variations in water depth if the tidal range is great. No corrections of this nature have been made to the series of observations in this report, since the amplitude of swell is not studied in detail.

It remains to determine how often records need to be made, and what the duration of each record shall be. It is inadvisable to work with records of too long duration, for theory shows that the wave trains arriving from a storm are slowly changing their period, and, indeed, it is this trend which is to be revealed by the series of records. On the other hand, the estimates of period can only be crude if the records are very short, for broadly speaking it is only possible to measure the periodicity of the various wave trains to the nearest whole number of cycles which they exhibit in the length of the record. There is an optimum duration for the record and appendix 3 shows that this duration varies between about 17 and 45 min. depending on the distance and bearing of the storm and on the tidal streams. It is also shown that for the close study of the change in period of bands of swell it would be desirable to record continuously, and to analyze consecutive sections of the record, but it was not found practicable to take records more frequently than once every 2 hr.

Wave recording

Most observations on waves have been made visually, the chief exceptions being measurements made using a float recorder which moved in vertical guides attached to a pier at the Scripps Institution of Oceanography, and stereophotographic measurements made from German and Japanese vessels. The Germans also used a pressure recorder suspended from a floating buoy when studying the landing behaviour of seaplanes. More practical methods were developed by Admiralty scientists when reliable information was needed for military purposes.

The measurements taken in the year 1945 were made with an American powerphone, an inductive type of hydrophone, modified in H.M.S. *Vernon* to make the response more linear and to reduce hysteresis in the diaphragm. The instrument is laid on the sea bottom, and the deflexions of the diagram due to the wave pressures produce changes in flux in a coil which is connected by submarine cable to a recording fluxmeter on shore. The instrument was laid in a depth of 110 ft. of water off Pendeen in north Cornwall. In September 1945 two additional instruments of different construction were laid off Perranporth, one in 70 ft. of water and the other in 35 ft. These instruments had an elastic member consisting of a small metal bellows, whose deflexions were converted into an electric current by a bridge circuit, and recorded by a galvanometer on shore. This instrument is made by the Cambridge Instrument Company. Two inverted echo-sounding instruments were laid in the same positions as the pressure recorders to record echo-profiles of the surface.

Analysis of wave records

It was soon established that the trend of wave periods could not be followed with any certainty by visual observation of the waves if only for the reason that they are difficult to make, especially at night. Visual examinations of the wave records provided by an automatic recording instrument were also disappointing. Attempts were made to estimate the component bands of periods by measuring the time interval between successive peaks on the record and presenting these measurements as a histogram. At favourable times these diagrams revealed the presence of an isolated band of swell; but generally they showed no significant trends, and it was concluded that they can give no more than a crude version of the frequency spectrum and it was not thought worth while to continue them. An analyzer capable of resolving a wave record in sufficient detail was constructed and first used in February 1945, and analyses prepared by it are the basis of the experimental results in the present article. The photographic records are obtained in the form of a black trace of variable area on white paper, this being achieved by arranging the mirror of the galvanometer or fluxmeter to throw on to the photographic paper in a slit camera, not a spot of light but a line of light which blackens one side of the record leaving the recorded curve as the boundary between the black and white parts. A time-trace is marked along one edge of the paper record by a light beam which is interrupted at intervals of 20 sec. by a clock-operated switch. The record is fastened round the circumference of a wheel which rotates about a horizontal axis carrying the record past an optical system which throws upon the record a horizontal line of light. The reflected light illuminates light-sensitive cells whose electrical output is, therefore, a continued repetition of the curve on the record. This electrical output is amplified and made to drive a vibration galvanometer. It is clear that if there is a component in the record having N complete cycles in the peripheral length of the wheel, this will produce a resonance of the galvanometer at its natural frequency of p cyc./sec. when the wheel is rotating at a speed of p/N rev./sec. The wheel is made to revolve at a speed which gradually decreases from a high value and the vibration galvanometer performs a series of transient resonances, one for each periodicity in the record. The resonances of the vibration galvanometer are converted to an electrical signal which drives a pen recorder, and the curve drawn by this pen is a series of peaks which constitute a Fourier amplitude spectrum of the curve on the record. A second pen operated by a second optical system and selective circuit working on the barred time-trace, draws a series of peaks which are the frequency analysis of the time-trace and serve as a period scale for the analysis.

The operation involves a continued repetition of the wave record, and the analyzer resolves it into a series of components which have a whole number of cycles in the length of the record, and whose periods are, therefore, submultiples of the interval of time represented by the record. The analysis is a series of peaks at these periods, but they are sufficiently numerous to present an adequate picture of the true frequency spectrum; the application of such a method of analysis to a record of finite length is discussed in appendix 2. The analyzer is described in greater detail in *Nature*, **158**, 329 (1946).

Measurements of wave pressure have been chosen for analysis in preference to measurements of surface elevation made by an inverted echo-sounder because the apparatus measuring wave pressure is more suitable for automatic operation at predetermined hours, and it is easier to arrange for it to provide a black and white record suitable for harmonic analysis. Further, it is shown in appendix 1 that the pressure fluctuation is nearly sinusoidal, so that pressure records are more suitable for harmonic analysis. Most of the work has been done with records of 20 min. duration taken every 2 hr.

Wave spectra

Typical wave spectra are reproduced in figures 2, 9 and 12, and each is marked with the date and time of the recording and a period scale in seconds. They show the tendency of the machine to resolve the spectrum into a large number of discrete frequencies, represented by individual peaks on the pen record. Some idea of the scale of the pressure amplitude is given by including the height, in inches of water, of the maximum pressure fluctuation in the record. For the interpretation of the pressure amplitudes which are measured in inches of water at a mean depth of 110 ft. at Pendeen and 70 ft. at Perranporth, it has been mentioned that the pressure at the bottom of the sea of depth h, due to a train of waves of amplitude a is $a\,\mathrm{sech}\,(2\pi h/\lambda)$, where λ is the wave-length, the pressure due to short waves being less than that due to long waves of the same height.

It is not intended to attempt a precise study of the amplitudes of wave components in this report, which is mainly concerned with the presence of particular frequencies. For this purpose it is the general outline of the spectrum that is significant; the occurrence in the spectrum of a frequency band which stands substantially above the background and appears in a number of consecutive spectra shows the presence of waves of that period. Defects in the photographic records and analyzing technique lead to a more or less obvious background in the spectra, but there is generally no difficulty in distinguishing such spurious indications appearing in individual spectra, from the genuine frequency bands that appear in a succession of spectra.

An almost unbroken series of wave-pressure records and frequency spectra has been obtained since February 1945, and it was soon evident that frequency bands could be distinguished and attributed without any uncertainty to particular storms shown in the relevant meteorological charts. In particular, the frequency analysis made it possible to ascertain the time of arrival of particular wave periods to within about 2 hr.

From this stage the best method of checking the theoretical conclusions seemed to be to select incidents of a sufficiently simple nature and obvious interpretation. Such a method is the nearest approach that can be made to controlling the natural conditions to allow precise conclusions such as might be drawn from controlled experiments in a laboratory. It was considered that attempts to solve the more complex problems by statistical treatment of the data should wait till account could be taken of such simple rules and reasonable assumptions as might result from the detailed study of selected incidents.

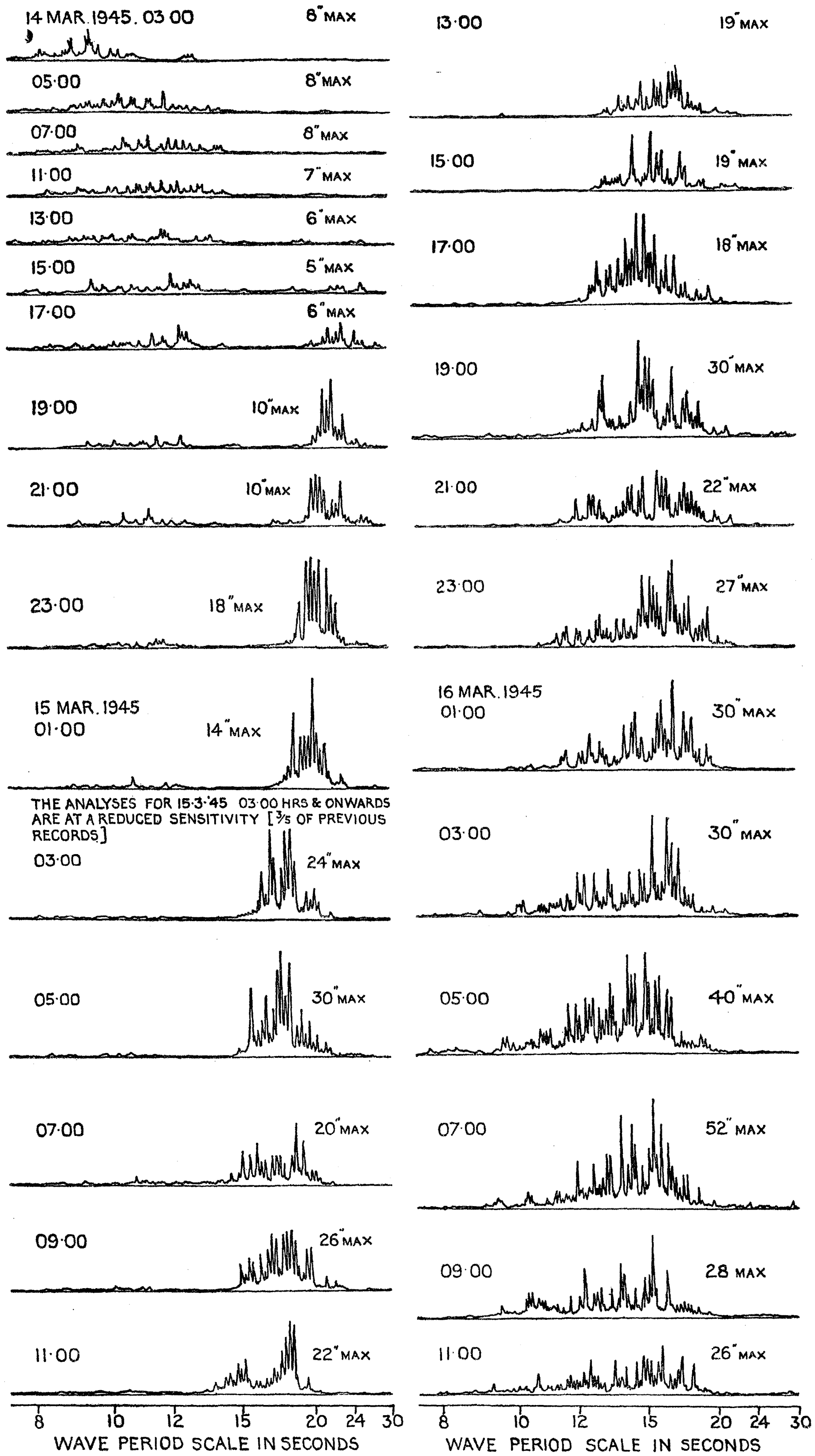

FIGURE 2. Wave spectra at Pendeen 14 to 16 March 1945.

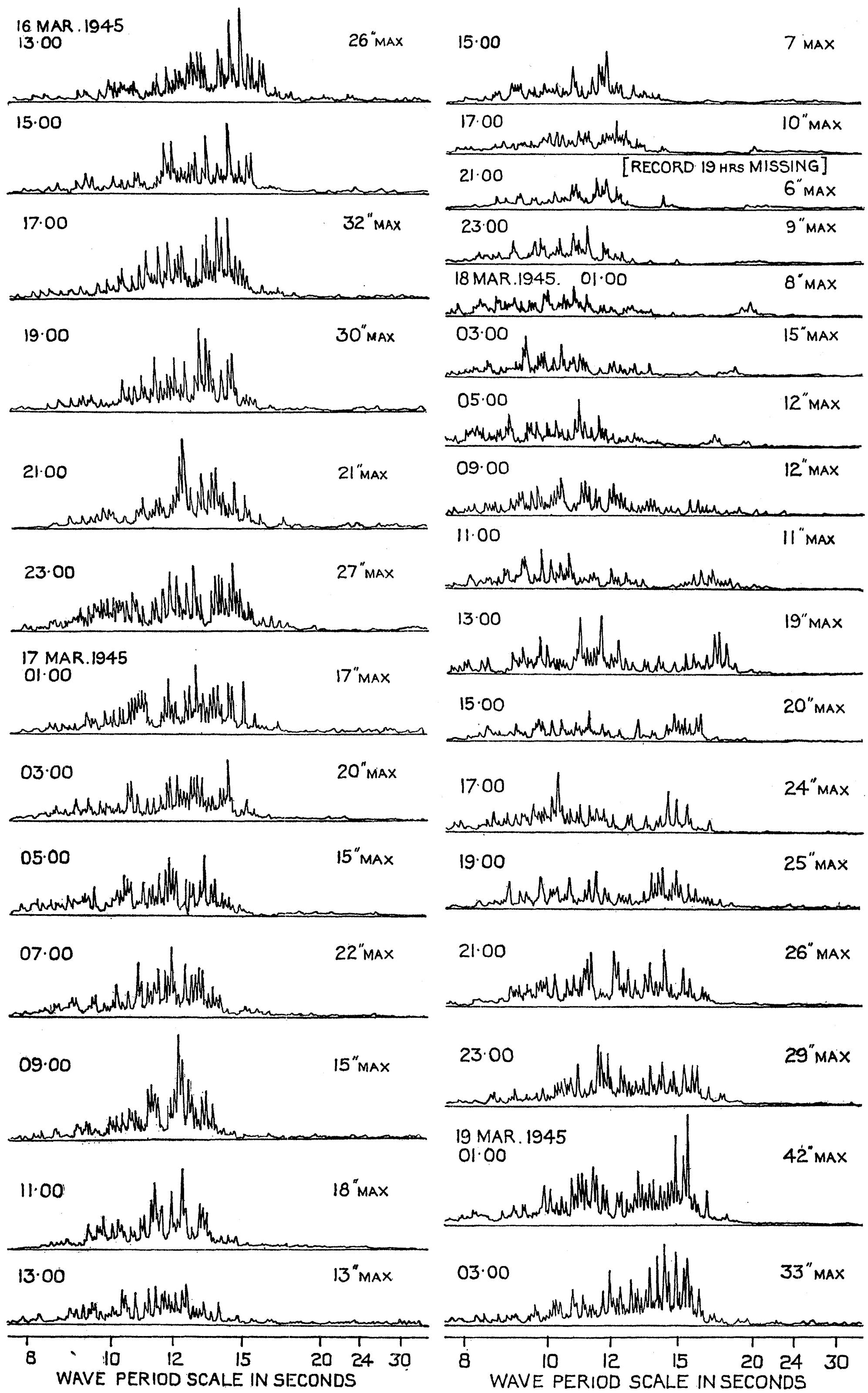

FIGURE 3. Wave spectra at Pendeen 16 to 18 March 1945.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

(*a*) 14 *to* 18 *March* 1945

The first few spectra in figure 2 show only waves of period less than 14 sec., whose amplitudes decrease in successive spectra. Some activity can be seen in the region of 18 to 23 sec. at 13.00 hr. 14 March, and in the following analyses it becomes more marked. Its persistence above the background, and increase in amplitude, show that it marks the arrival of long-period swell. As time goes on the mean period of the band decreases, showing that the long swell is followed by shorter swell. After the morning of 16 March, in figure 3, the amplitude also decreases, and by the afternoon of 17 March the spectra resemble those at the start, which themselves represent the residue to a previous swell. The long swell arriving on 14 March can be shown to have its origin in the mid-Atlantic storm which appears at its most intense development in the meteorological chart reproduced in figure 4.

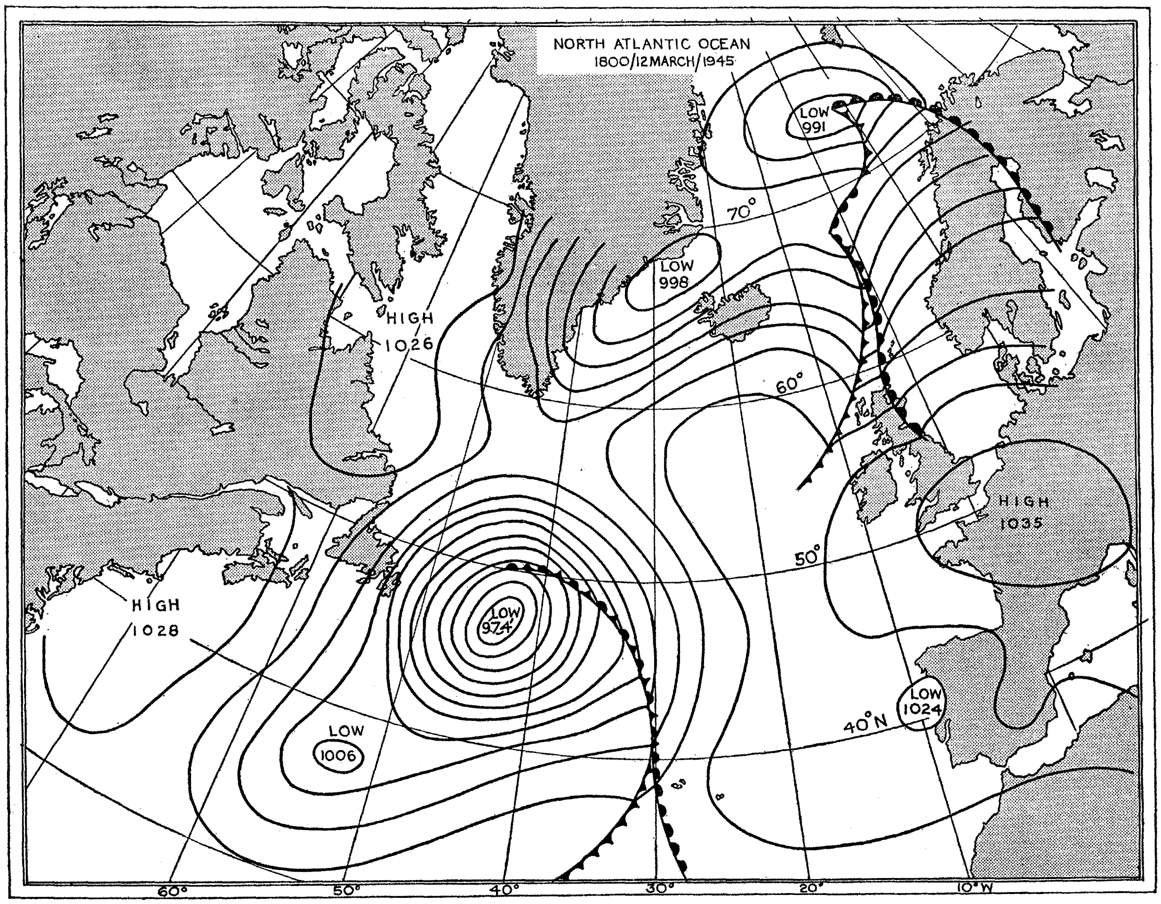

FIGURE 4. Meteorological chart of the North Atlantic Ocean 18.00 hr. 12 March 1945.

To study the swell from this storm a propagation diagram has been constructed in figure 5. The figures represent the maximum reported wind strength, on the Beaufort Scale, at all distances up to 3000 miles from Pendeen between 11 and 15 March, but only include winds whose direction is not included at more than 90° to the direction of Pendeen. Contour lines are drawn round the observations greater than force 8. Propagation lines are drawn, according to the method described on p. 530, for the maximum and minimum wave periods arriving at different times on 14 and 15 March, broken lines being used for the maximum periods and full lines for the minimum periods; to avoid confusion the number of lines shown

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

in the diagram has been restricted to the minimum sufficient to give a representative picture. The velocities used are the theoretical group velocities for deep water, for it is shown in appendix 4 that the shoaling of the water in the coastal region will have a negligible effect on the total travel time.

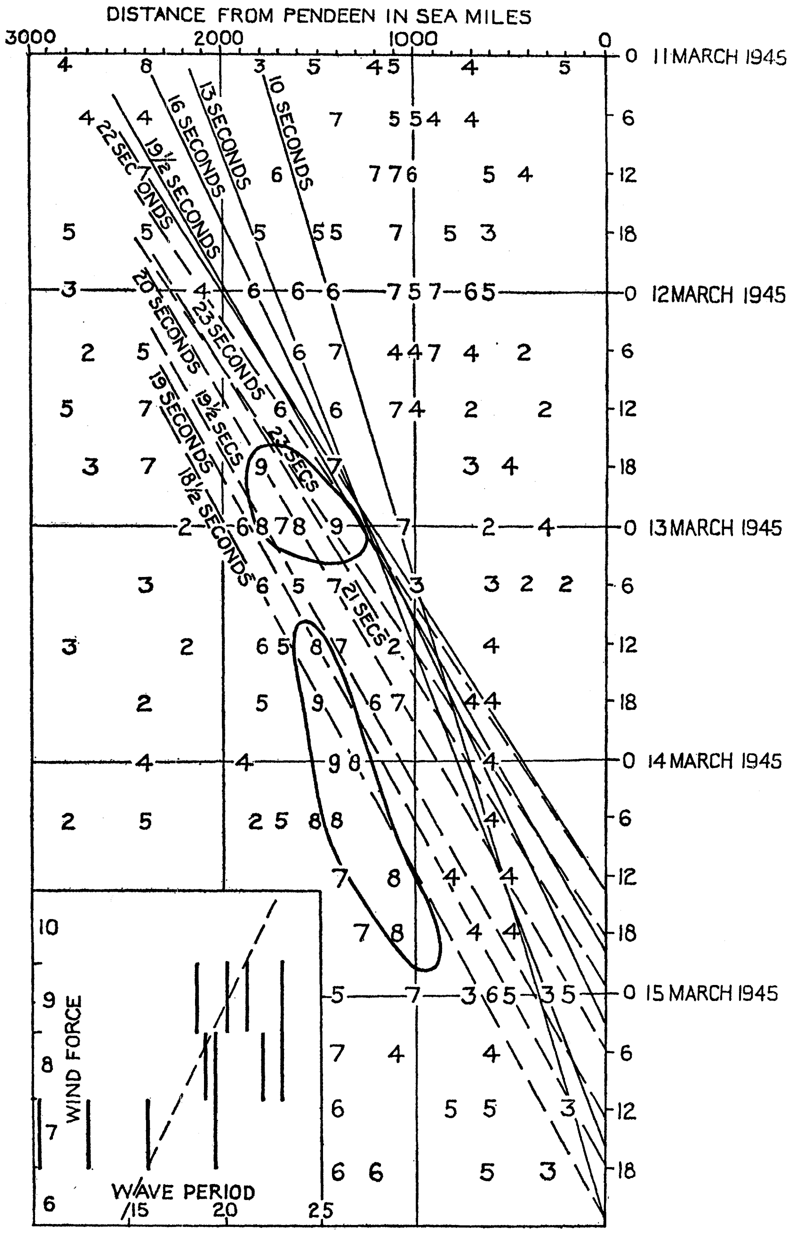

FIGURE 5. Propagation diagram 11 to 16 March 1945, using reported winds blowing within 90° of a line to the recording station.

The propagation lines for the minimum wave periods pass through the early part of the storm, in which the wind is rising, while those for the maximum periods pass through the later stages in which the wind has reached force 8 and 9, and there is good reason to believe that this is the order in which the short and long periods are generated, the trains of swell of maximum period being generated when the wind is strongest. Some further justification is provided by the comparison between maximum reported wind forces and maximum

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

recorded wave periods which is made in figure 6. The data from thirteen storms for which there are plenty of wind reports are used; the reported wind forces are represented by lines instead of points because the unit on the Beaufort Scale covers a range of 5 to 6 knots at this part of the scale. When ships in the same neighbourhood have sent reports differing by one unit the average is plotted. The distance of each storm from the recording station is noted against the entry, and it will be seen that except for the local storm there is good correlation between the maximum wind force and maximum swell period between periods of 16·5 and 20 sec. Periods greater than 20 sec. appear to be generated without any increase in wind force, but it is possible that this unexpected result may be due to errors in wind reporting; there may be some difficulty in distinguishing between the Beaufort numbers greater than 9 and some reluctance, through inexperience or fear of exaggeration, to report the higher numbers. No satisfactory explanation can be suggested for the anomalous relation between the maximum wave period and maximum wind for the local storm, but except for this observation the distance travelled by the swell appears to have no bearing on the relation.

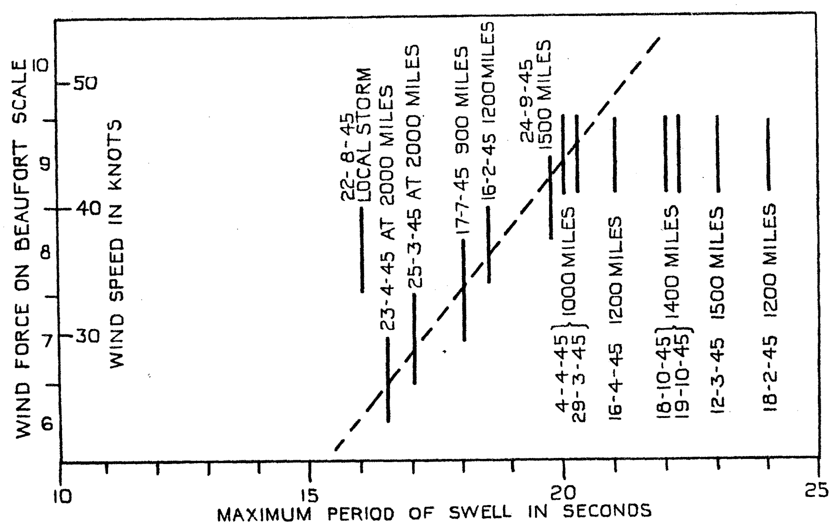

FIGURE 6. Correlation between the maximum wind strength and the maximum period of swell generated for thirteen storms.

We should, therefore, expect that the swell appropriate to each propagation line in figure 4 has originated in the area of highest wind which the propagation line encounters, and that the relation between the period of the swell and the highest Beaufort number on each line should be the same as the relation shown in figure 6. The data are plotted in the inset in figure 5, and the relation between maximum wave period and maximum wind strength from figure 6 is indicated by a broken line. The correlation is not good, and the dispersion of the data emphasizes that although the general disposition of the propagation lines about the storm area in figure 5 is reasonably good, the agreement is not perfect in detail.

To attempt a more precise study it is reasonable to give more weight to winds which blow towards the recording station since the waves travel mainly in the direction of the wind, and a new diagram was drawn taking account only of winds blowing within 30° of the direction towards the recording station. A further refinement introduced is the use of wind estimates based on the pressure distribution instead of the wind force as reported by ships.

Although it is questionable whether such an estimate for a particular place and time is better than a ship's report, the method offers some advantages where there are not many reports; and it smooths out such irregularities as are introduced by fluctuations in wind strength and faulty reporting. The calculation of the surface wind strength from the spacing of the isobars was done with the aid of a Sea Surface-Wind Scale used by the Naval Meteorological

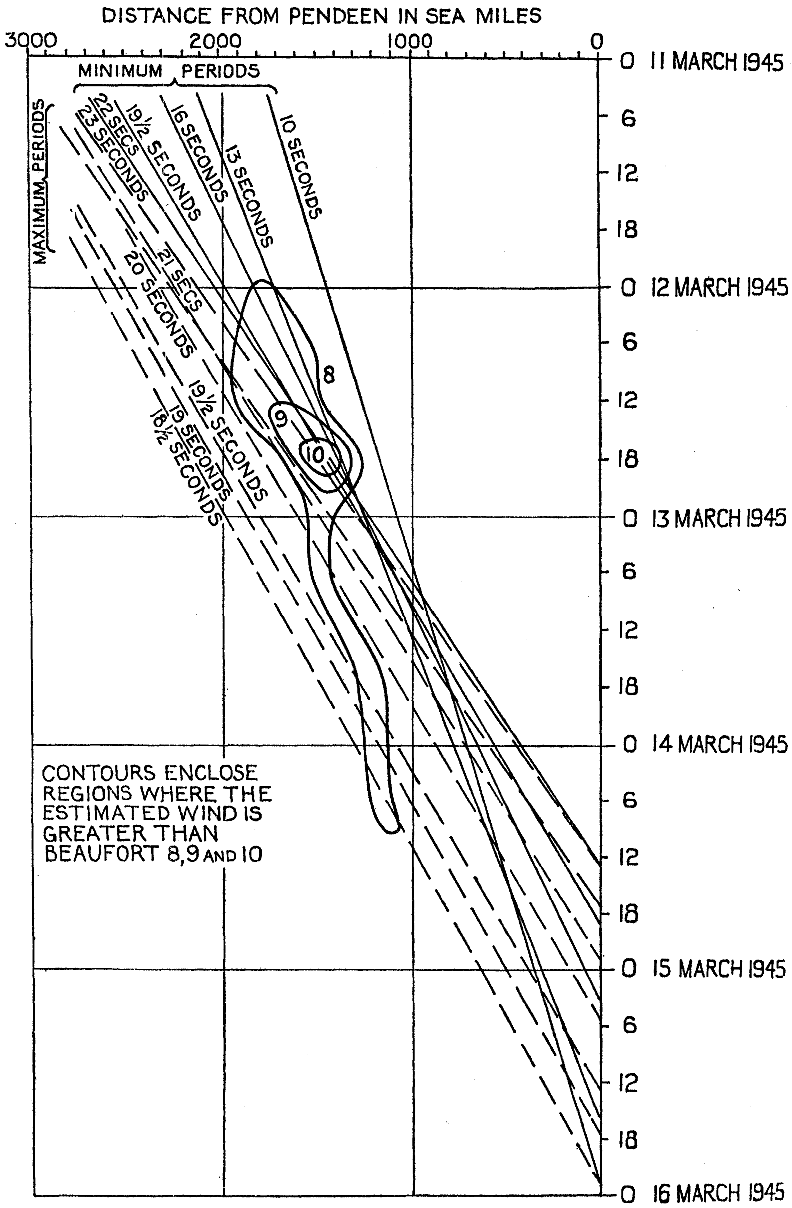

FIGURE 7. Propagation diagram 11 to 16 March 1945, using winds estimated from the pressure distribution, blowing within 30° of a line to the recording station.

Branch, and it was assumed that the surface wind was inclined at an angle of 15° to the isobars. The new propagation diagram based on these estimates is reproduced in figure 7. Contour lines are drawn round areas in which the wind force is 8, 9 and 10, and the propagation lines are the same as those in figure 5. The strongest winds occur somewhat earlier and nearer to the recording station than they appeared to do in figure 5, and the disposition of the propagation lines about the storm is improved in detail. The propagation lines for the

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

minimum periods pass through the early part of the storm area, the period increasing as the intensity of the wind increases. The lines for the maximum periods pass through the later part of the storm area and the period decreases as the intensity of the wind decreases.

The satisfactory way in which the propagation lines join the measured times at which the different periods forming the upper and lower limits of the wave band cease to arrive at the recording station, and the times at which they can be supposed to begin and cease being generated in the storm area, suggests that the assumption underlying the propagation lines, that the waves travel with the theoretical group velocities appropriate to their periods, is reasonably accurate.

This way of treating a storm may be a suitable way of predicting the arrival of swell by an examination of the isobaric charts of the ocean once the rules of generation and propagation have been established, but the assumptions that have been made about the direction of significant winds make it of less value for demonstrating the fundamental rules of the propagation of swell. The area of the storm in this example is too great, and it is too near the measuring station for it to serve as anything more than an approximate means of measuring the rate of travel of swell.

(*b*) 24 *June* to 2 *July* 1945

With the experience gained from the study of the storm of 14 to 18 March it was evident that the best conditions for testing the assumption of linearity of propagation and the classical group velocity would be obtained by examining the swell from a small intense storm at a great distance, and a search through the records showed a suitable tropical storm which developed over Florida and reached its greatest intensity off the coast between Cape Hatteras and Nantucket on 26 to 27 June. Sections of meteorological charts showing four stages before, during and after the period of greatest intensity are shown in figure 8. The very slow movement of the storm is an advantage, because the waves are able to run quickly out of the storm area.

The wave spectra obtained by the analysis of records taken at Pendeen between 30 June and 2 July are reproduced in figure 9. The swell from the tropical storm was first detected at 19.00 hr. on 30 June, when it appears as slight activity in the region of 18 sec. period. The frequency band increases in amplitude in the subsequent spectra and its mean period decreases to 13 to 15 sec. by 17.00 hr. on 2 July.

Close examination of the spectra shows that the decrease in period is not uniform, and to illustrate this irregularity the maximum and minimum periods which limit the width of the band in successive spectra are plotted in figure 10. Both limits of the frequency band show a trend towards lower periods, but exhibit irregularities about the smooth curves greater than can be accounted for by the uncertainty in estimating the periods from the spectra. It will be shown later that they can be attributed to the effect of the passage of the waves through tidal streams near the recorder, and that it is justifiable to use a smoothed curve for studying the propagation in the open sea.

A propagation diagram with lines drawn for 12 hr. intervals on 1 and 2 July is reproduced in figure 11. The path of the storm is indicated by dotted lines, and the numbers inside the lines are the wind forces on the Beaufort Scale reported by ships in the storm. The reported winds were not always directed within 90° of Pendeen, but in such a small circular storm it

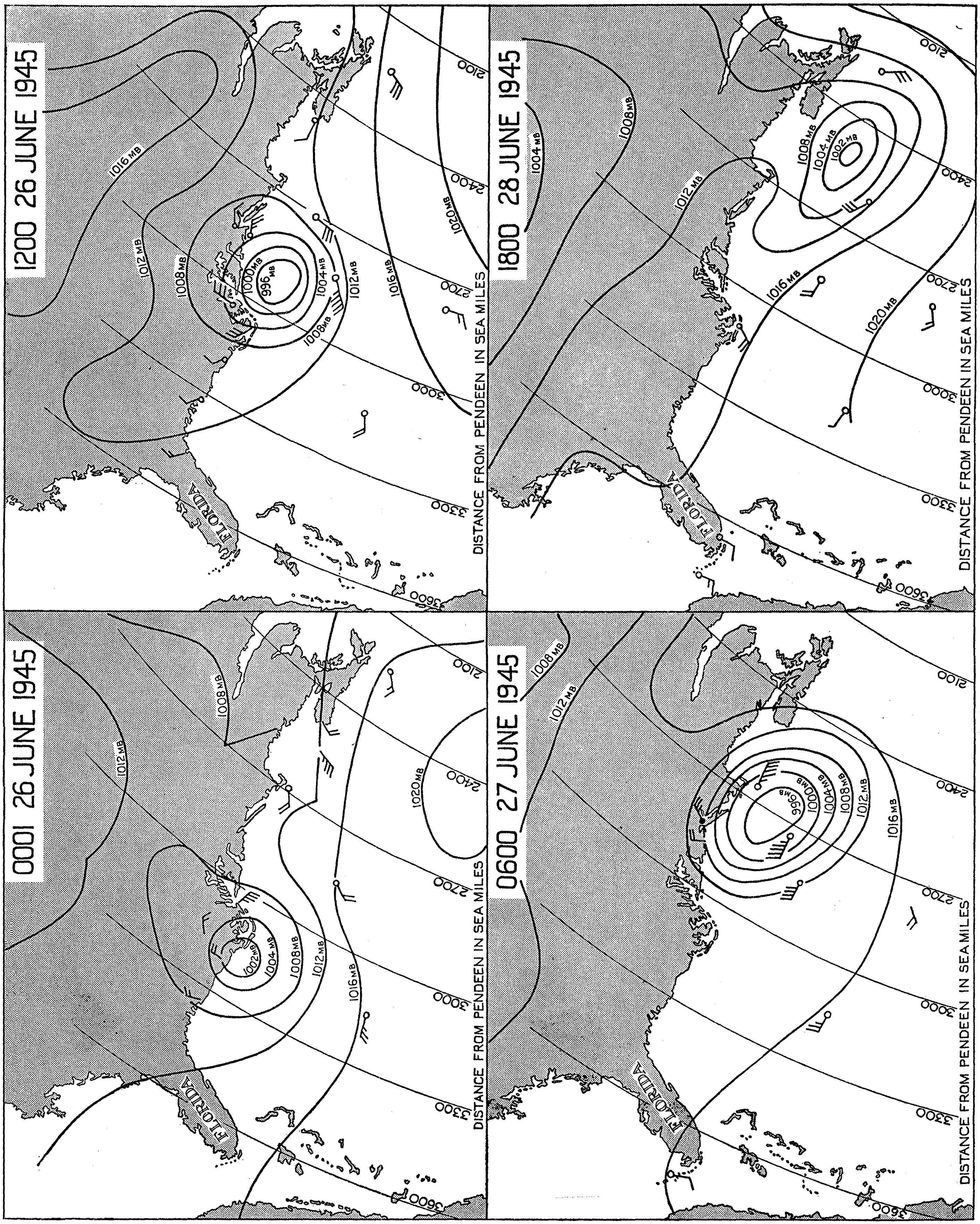

FIGURE 8. Meteorological charts of the U.S. Atlantic coast 26 to 28 June 1945.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

is reasonable to suppose that the strength of the wind directed to Pendeen was approximately the same as that reported for any other direction. The propagation lines for the maximum periods are represented by broken lines and those for the minimum periods by full lines. The wave periods limiting the width of the frequency band are taken from figure 10.

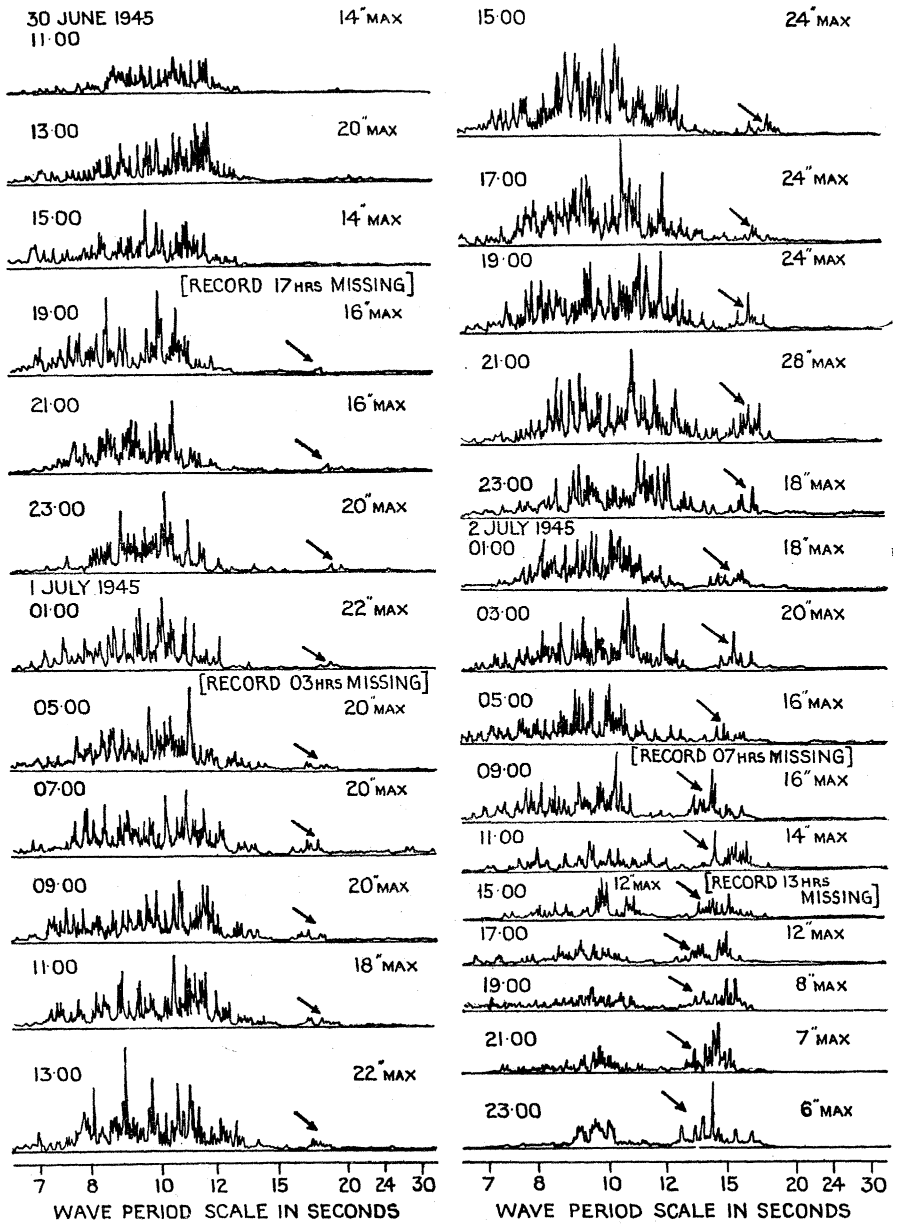

FIGURE 9. Wave spectra at Pendeen 30 June to 2 July 1945.

The propagation lines are very reasonably distributed about the area of greatest wind strength, indicating that the shortest periods are generated first, while the wind is rising, and the longest periods when the wind is strongest. As the strength of the wind decreases, and the distance over which the strong wind acts on the waves decreases, the longer periods cease to be generated, and the upper limit of the frequency band falls from 19 to 16 sec. As the shortest periods generated in the early part of the storm travel from the storm area they are overtaken by the longer periods, so that the propagation lines for all periods up to

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

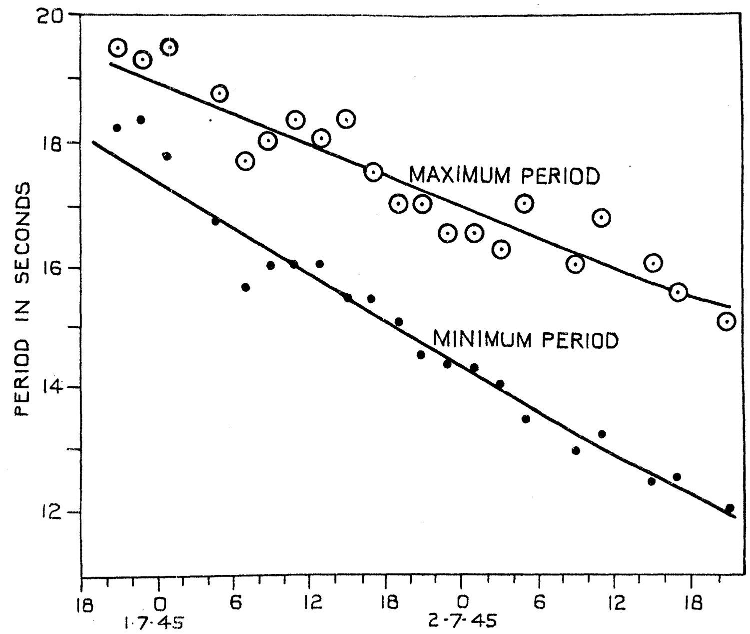

Figure 10. The maximum and minimum periods limiting the frequency band from the tropical storm of 26 to 28 June in the wave spectra of 1 to 2 July.

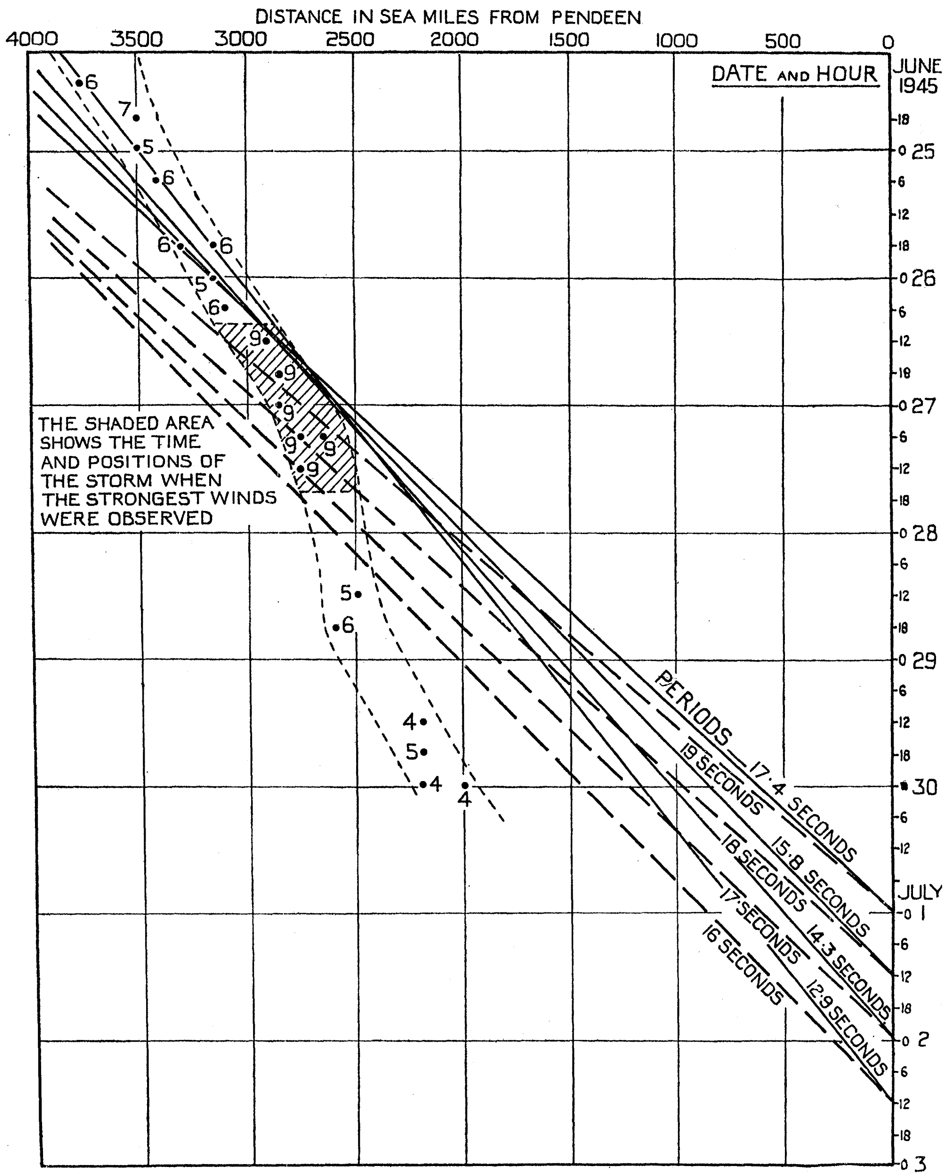

Figure 11. Propagation diagram 24 June to 2 July.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

the maximum cross each other between the storm and the coast. The propagation lines for the periods which form the upper limit of the frequency band in the spectra do not intersect, since their times of arrival at Pendeen are determined by the times at which they cease to be formed in the storm area; the longest waves cease to be generated first, and they are the first to disappear from the spectra at Pendeen.

As in figure 7 the propagation lines are shown to represent the classical group velocity appropriate to each wave period, and since they give a good representation of the travel of the swell it can be concluded that the theoretical group velocities are applicable. Because of the difficulty in estimating the time at which the waves of a certain period might be generated or cease to be generated, the comparison of the observed and theoretical velocities is not very precise, but it seems reasonable to conclude that an alteration of $\pm 5\%$ in the slope of the propagation lines in figures 7 and 11 would not be justified. It will be seen that this example is not invalidated by assumptions regarding the directions of winds which generate the observed swell.

(*c*) 14 to 19 *May* 1946

Some forty examples of the development of swell have been examined in sufficient detail to show that the time of arrival of the swell is in fair agreement with theory, and to demonstrate that any distinctive wave band occurring in the spectra can be traced to the storm in which it was generated if the relevant meteorological charts are available. In almost every instance it has been found that such bands have their origin in storms in the North Atlantic Ocean, but on at least two occasions there is unmistakable evidence that the swell has been generated in storms occurring beyond the limits of this ocean.

Figures 12 and 13 show the wave spectra obtained by analysis of records made every 2 hr. at Perranporth between 14 and 19 May 1946. The first few spectra in figure 12 show the gradual decay of waves of less than 10 sec. period that were caused by a local wind, but after 11.00 hr. on 14 May two new frequency bands develop, one between 10 and 12 sec., and the other at about 21 sec. The first can be traced to a storm which developed south of Newfoundland on 8 to 9 May, and will not be considered in detail, but the second deserves special attention. The most remarkable feature is its very slow decrease in mean period in successive spectra, from 21 sec. at 11.00 hr. on 14 May to 15 sec. at 05.00 hr. on 19 May; the swell of 10 to 12 sec. period arriving at this time belongs to a separate frequency band which first arrived at approximately 23.00 hr. on 17 May with a mean period of 14 to 15 sec. and moved away from the swell which had persisted since 14 May.

As soon as the slow decrease in mean period of the persistent frequency band is studied in detail, it is apparent that the trend shown by successive spectra is not regular. In addition to the trend towards lower periods the upper and lower limits of the band in successive spectra show an oscillation of approximately 2 sec. amplitude and 12 hr. period. This is shown very clearly by figure 14, in which the maximum and minimum periods are plotted against the time at which the spectrum was examined. The approximation of the period to the semi-diurnal tidal period immediately suggests that the wave periods are affected by the tidal streams.

It can be shown theoretically that a tidal stream must change the periods of waves relative to a stationary observer: if the waves enter an area at slack water with velocity v and the whole

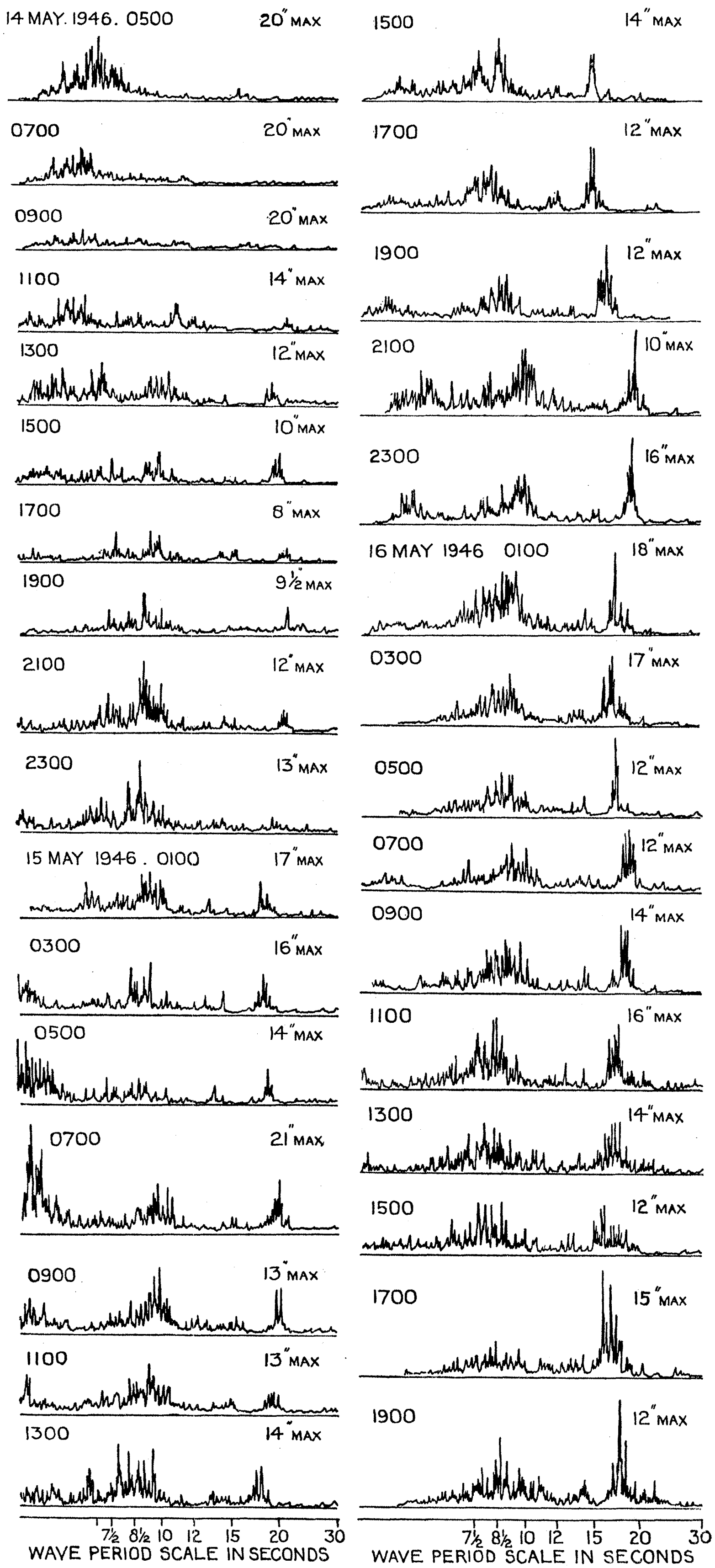

FIGURE 12. Wave spectra at Perranporth 14 to 16 May 1945.

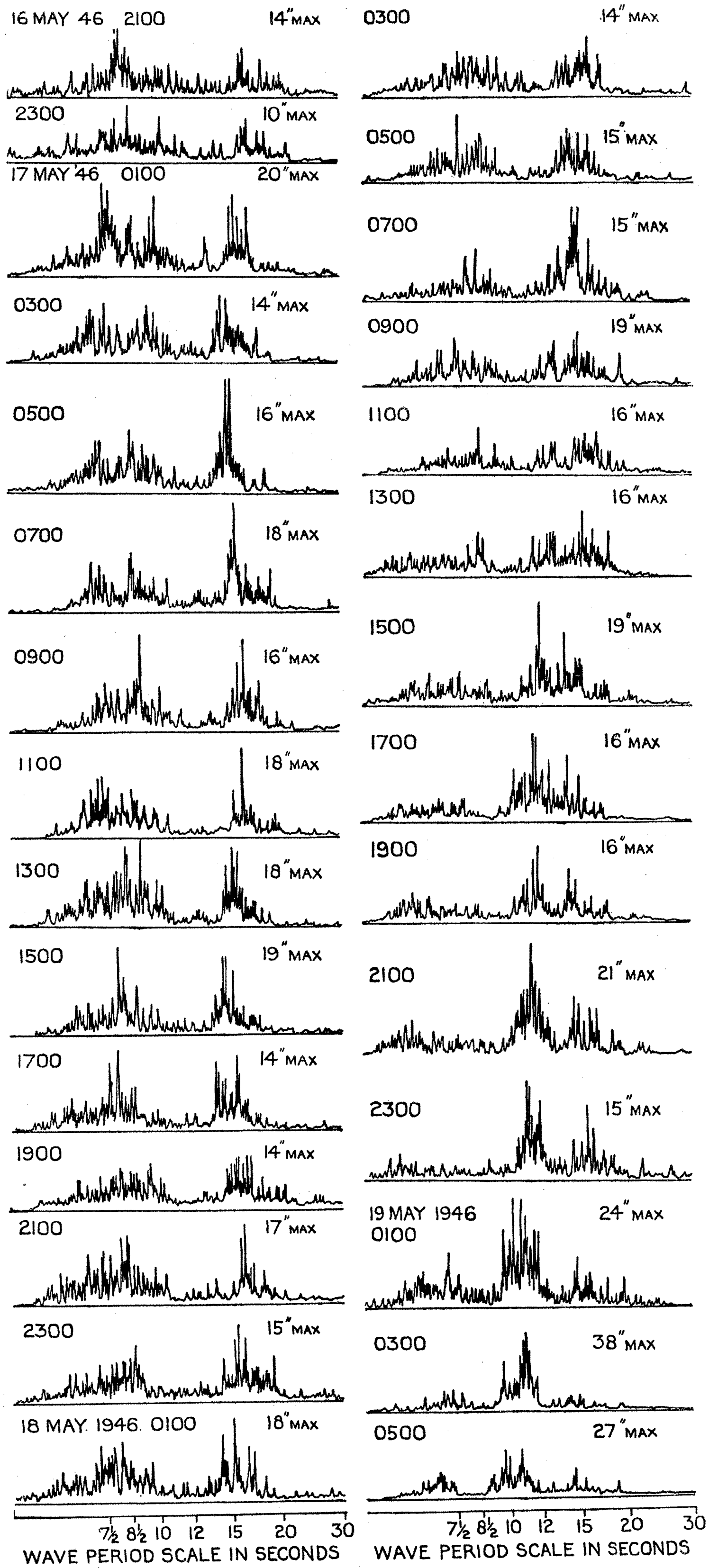

FIGURE 13. Wave spectra at Perranporth 16 to 19 May 1945.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

area begins the move with velocity u, then the velocity of the waves relative to a stationary observer will be $v+u$, and the wave period appears to be reduced by the factor $v/(v+u)$. The apparent reduction in period will be greatest if the waves enter the area at the time of maximum opposing stream, and pass the recording instrument 6 hr. later, when the stream has its greatest velocity in the same direction as the waves. Under these conditions u will be the algebraic difference of the velocities of the ebb and flood streams.

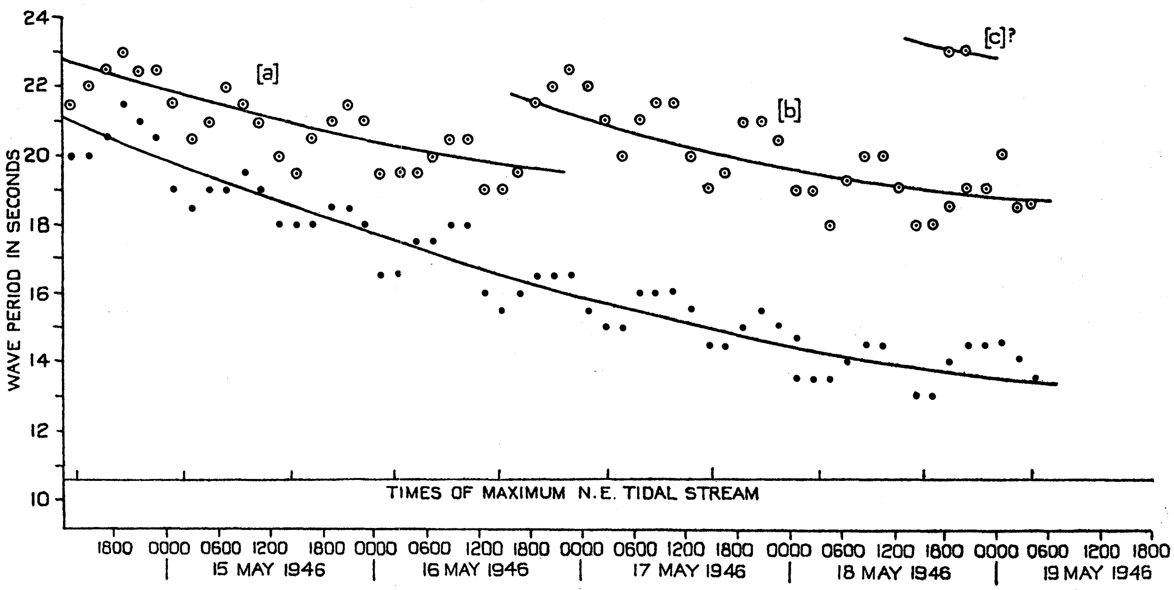

FIGURE 14. The maximum and minimum periods limiting the 23 to 14 sec. frequency band in the wave spectra of 14 to 19 May 1945.

There is good reason to believe that the narrow band of swell arriving between 14 and 19 May was travelling from the south-west, and since its group velocity was approximately 30 knots it would take approximately 6 hr. to travel from the 100-fathom line to the recording station. It is only within the 100-fathom line that the tidal streams are likely to be appreciable, and the closest estimate that can be made is that the streams set north-east and south-west with a maximum rate of approximately 1 knot. Thus the overall change can be regarded as 2 knots, and if the swell crosses the 100-fathom line when the south-west stream is strongest, the wave velocity, which is 60 knots, will be changed at Perranporth by 2 parts in 60, and the wave period will be reduced by two-thirds of a second. This agrees reasonably well with the variation in period about the smooth curve in figure 14, which suggests that the period is alternately decreased and increased by approximately 1 sec. about a mean value. There is also agreement in phase because the greatest reductions in period occur when the tidal stream at the recording station is running fastest to the north-east. These times are shown in figure 14 by a row of marks above the time scale. The evidence appears therefore to justify the conclusion that the fluctuations in period are due to the tidal streams, and to justify the use of a mean curve for studying the wave travel in deep water. It is believed that the irregularities shown by the maximum and minimum wave periods in figure 6 are caused in the same way, although they do not give such a clear indication of the semi-diurnal period.

There is no difficulty in drawing a smooth curve through the minimum periods in figure 14, but there appears to be at least one significant discontinuity in the trend of maximum periods; after 17.00 hr. on 16 May the spectra suggest that the frequency band is widened by the arrival of swell whose maximum period is 2 to 3 sec. longer than the mean maximum of the previous spectra. A second widening at the upper end of the frequency band possibly occurs at 19.00 hr. on 19 May when there is some, though not very reliable, indication of activity at periods up to 23 sec.

A propagation diagram for the period 4 to 19 May is reproduced in figure 15. To simplify the diagram as much as possible propagation lines are drawn only for the narrow frequency

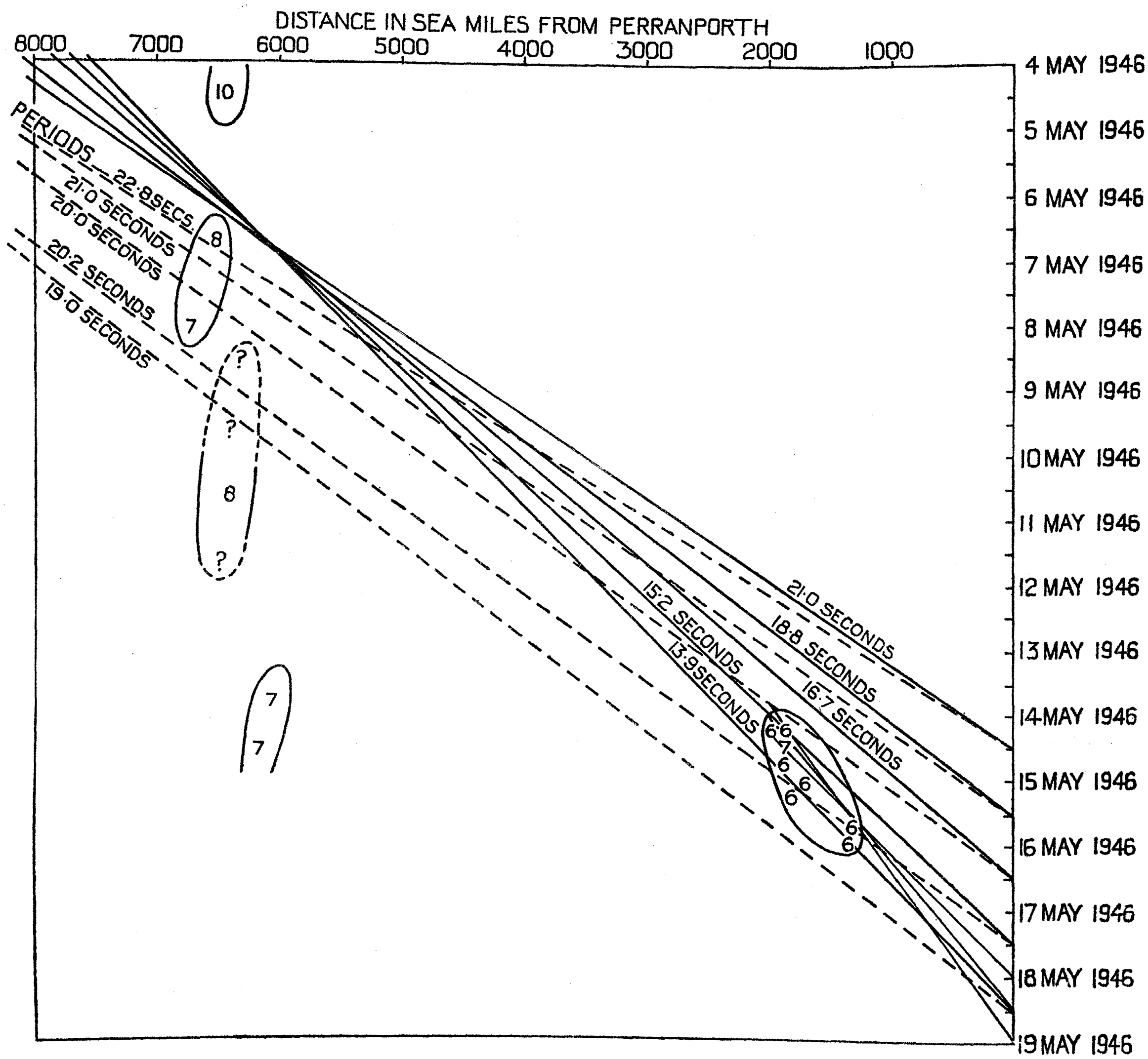

Figure 15. Propagation diagram 4 to 19 May 1946.

band which is being considered, and for the minimum periods of the subsidiary band whose mean period decreases from approximately 15 to 11 sec. on 18 May, sufficient to show that it was generated in an area 1300 to 2000 miles from the recording station, and not to be confused with the main band. The propagation lines for the minimum periods of the main band intersect at a distance of 6000 miles from the recording station, and this is regarded as unmistakable evidence that the swell was generated at a distance of 6000 to 7000 miles. The storm must, therefore, be sought in the Southern Ocean, probably in the Falkland sector to which there is an open great circle course from Land's End. The meteorological

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

information that is available from this region is shown in the propagation diagram; it is considered too scanty to justify an attempt to find detailed relationships between the wind reports and the propagation lines. It is believed that winds of force 9 would be sufficient to generate the wave periods that were recorded, and since south-west winds of this force occur very frequently in the Falkland sector, it is reasonable to suppose that trains of swell similar to that recorded between 14 and 19 May are often propagated towards the British Isles. When the recordings were made the wave-recorders were operating at their highest sensitivity, because the sea was very calm; if it were normally possible to run the instruments at this high sensitivity Southern Ocean swell might be detected more often. It is also possible that the winds and other wave and swell systems, particularly those in the Trade Wind regions, were more favourable to the passage of the Southern Ocean swell. The question is being investigated.

The maximum height of the Southern Ocean swell recorded at Perranporth was approximately 12 in.

Conclusions

The good agreement between the distribution of the propagation lines based on the measured time of arrival of successive wave periods and the time at which these periods begin and cease to be generated in the storm area, found on the data of 14 to 18 March and 24 June to 2 July 1945 is representative of all the wave spectra and storms for which adequate meteorological data are available, and it justifies the assumptions that have been made:

(*a*) Propagation is linear, so that trains of swell of different length and period behave independently of each other.

(*b*) Waves are propagated with the classical group velocity appropriate to their period.

(*c*) There is an upper limit to the wave period generated by a wind of given strength which is an increasing function of the wind strength.

The third assumption has been studied less critically than the others, and it will probably be necessary to add some qualifications about the duration of the wind and its fetch. Waves appear longer at the end of the fetch than at the beginning, but the observation should be checked by frequency analysis to find whether this is not due to a change in distribution of energy in the spectrum rather than to a change in the value of the longest period in the spectrum.

The confidence with which a frequency band that appears in the wave and swell records off the north coast of Cornwall can be traced to its origin in a local or distant storm greatly increases the value of such records for the study of the relations between the strength, duration and fetch, of the wind and the wave characteristics. The study of the attenuation of waves as they travel towards a distant coast through calm water, favourable or opposing winds is also facilitated.

The constant help of the Naval Meteorological Branch and the Admiralty Swell Forecasting Section, especially in supplying meteorological charts for the whole of the North Atlantic Ocean since February 1945 is gratefully acknowledged. The authors also wish to express their indebtedness to other members of their research group at the Admiralty Research Laboratory. The development of the underwater pressure-recorder is largely the work of C. H. Mortimer; the electronics of the first model of the analyzer were made by J. Darbyshire who also made the harmonic analyses up to October 1945 and on some later occasions. Most of the analyses after this date were made by W. M. Cotsworth, who is in charge of the recording station at Perranporth, using a model of the analyzer designed and constructed by M. J. Tucker, to whom is due most of the reliability and convenience of the analyzer in its present form. The design of the analyzer has been facilitated by a robust vibration galvanometer made by G. Collins. An analyzing wheel with a power drive and a truly exponential decrease of speed, constructed to the designs of F. E. Pierce, is now coming into use. The authors' thanks are due to M. S. Longuet-Higgins for a critical reading and discussion of the appendices, and to G. E. R. Deacon for considerable advice and help in the preparation of this article. The results are published by permission of the Admiralty.

Appendix 1. Pressure on the sea bed

If the frequency analysis of wave pressure is to be practicable, a regular train of waves travelling unchanged in form must give rise to a nearly sinusoidal fluctuation of pressure. Now the waves may be of finite height, introducing second and higher harmonics into the expression for the pressure. However, except for high waves in very shallow water, it is usually sufficient to consider the effect of the second harmonic. It has been shown by Miche (1944) that the pressure is to this order of approximation independent of the vorticity in the fluid, provided that the resultant shearing velocity is small compared with the particle velocity. We are, therefore, justified, when calculating the pressure, in assuming that the motion is irrotational, so that we can use the velocity potential given by Stokes (1847), to second order. Using this potential, we shall now show that the effect of the second harmonic is usually negligible on the sea bed where the pressure is to be measured, and hence that the pressure of a single wave-train is described with adequate accuracy by a single sine wave.

We shall write the potential in the modified form due to Struik (1926). In this form, the potential is expressed as a Fourier series:

$$\phi = \phi_1 + \phi_2 + \ldots + \phi_r + \ldots,$$

where $\phi_r = A_r(y)\sin(rkx - r\sigma t)$ is the rth harmonic constituent. The wave-length is $2\pi/k$, the period $2\pi/\sigma$. The solution holds for finite depth h. We shall compare the amplitudes of pressure appropriate to the fundamental and second harmonic. It is convenient to write the potential in terms of the wave height $2a$, using Struik's equation (37) (p. 629). The leading term in $A_1(y)$ is

$$\frac{ga\cosh ky}{\sigma\cosh kh}. \tag{A}$$

Bernoulli's theorem gives the pressure p at the sea bottom as

$$p = \rho\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\rho\left(\frac{\partial\phi}{\partial x}\right)^2,$$

where ρ is the density.

We consider first the harmonics in the linear term, then the harmonics in the squared term. The ratio

$$R = \frac{\text{amplitude of } \partial\phi_2/\partial t}{\text{amplitude of } \partial\phi_1/\partial t}$$

can be shown, by reference to Struik's formula, to be

$$R=\frac{3\mu\cosh 2kh}{\sinh^2 kh\cosh kh}$$

on the sea bed, where μ is a parameter defined by Struik. It can be shown that

$$\mu<\frac{a}{2h}\frac{kh}{\sinh kh}<\frac{a}{2h},$$

so that on the sea bed
$$R<\frac{3a}{2h}\frac{\cosh 2kh}{\cosh 2kh-1}\operatorname{sech} kh.$$

If the wave-length is less than ten times the depth,

$$R<\frac{3a}{h}.$$

If the wave height $2a$ is less than 6 ft. and the depth h is 100 ft., R is less than 0·1, which is negligible. The second harmonic can, therefore, be neglected in the consideration of long-period swell.

We now consider the quadratic term in Bernoulli's equation (B). The quadratic term introduces a double frequency, but it can easily be shown, using the term (A) obtained above, that its amplitude is small.

In fact
$$\frac{\text{amplitude of } \frac{1}{2}(\partial\phi_1/\partial x)^2}{\text{amplitude of } (\partial\phi_1/\partial t)}=\frac{a}{4h}\frac{kh}{\sinh kh}<\frac{a}{4h}.$$

For 6 ft. waves and a recorder at 100 ft. depth this ratio is 0·01.

Within these limits of error it can, therefore, be assumed that the pressure due to a single regular train of waves is sinusoidal on the sea bed and will be represented in the harmonic analysis by a single frequency.

The swell observed on the coast is not in general a regular train of waves, and many periodicities are present at the same time. There is no theory for a complex wave pattern, but it is expected that the theory discussed above will give at any rate the order of magnitude of the higher harmonics of each constituent wave train.

Appendix 2. Frequency analysis of pressure records

Introduction

We have seen that the velocity potential of swell may be described as the sum of a number of sine-wave potentials whose frequencies lie in well-defined and slowly varying ranges. The length of the records can be adjusted so that during any one record the frequency band from each generating area remains roughly constant. An analysis of the records according to their frequency characteristics should, therefore, provide material for the study of the generation, rate of travel, etc., of the swell.

However, the frequency analysis of only finite lengths of record presents serious theoretical difficulties. A well-known method of analysis is to represent the record as a Fourier series of sines or cosines whose coefficients are the integrals

$$a_s=\frac{1}{T}\int_{-T}^{T}p(t)\cos k_s t\,dt,\quad b_s=\frac{1}{T}\int_{-T}^{T}p(t)\sin k_s t\,dt.$$

Here $p(t)$ is the pressure amplitude, a function of the time t, given only in the finite time interval $-T<t<T$, and k_s takes the values $s\pi/T$ ($s = 0, \pm 1, \pm 2, \ldots$). This method has the advantage that the integrals are easily formed mechanically, by repetition of the record. But the method is certainly not unique, for the analysis could be made into either a sine series or a cosine series or both together. Nor is the set of Fourier frequencies the only possible set; we might take, for example, those determined by a transcendental equation of the type

$$k \tan k = a > 0.$$

A further difficulty is that in none of the above methods is a pure sine wave resolved into a frequency band of infinitesimal width unless it happens to be one of the fundamental frequencies of the method of analysis.

In this note we shall discuss what meaning, if any, should be attached to the harmonic analysis of a finite record $p(t)$, defined when $-T<t<T$, with special reference to the Fourier method of analysis, which is the most convenient to carry out in practice.

The continuation of a finite record

As the length of the record tends to infinity all the methods of analysis described above tend to the same process, that of taking the Fourier sine or cosine transform of an infinite record. Thus the Fourier transform of a record gives in a sense a unique representation of its frequency characteristics. But the Fourier transform is only defined when the record is of infinite length. We are, therefore, led to consider our finite record $p(t)$ as the section of an infinite record, continued beyond $(-T, T)$ by an arbitrary function of such a type as to ensure the existence of the Fourier transform.

Let $P(t)$ be such a function of t, equal to $p(t)$ in $(-T, T)$ and defined arbitrarily outside this interval. We shall call $P(t)$ a *continuation* of $p(t)$ outside $(-T, T)$. $P(t)$, being defined over the whole range $(-\infty, \infty)$, will have Fourier sine and cosine transforms $S(x)$ and $C(x)$. Consider first the cosine transforms $C(x)$ of all the possible continuations of $p(t)$. These will possess certain features in common, since they depend in part on $p(t)$. The common features of the $S(x)$ and of the $C(x)$ may be said to give all the available information about the frequency characteristics of $p(t)$.

We define a *masking function* $a(t)$ equal to unity in $(-T, T)$ and zero outside. Then consider the function

$$c(x) = \frac{1}{T}\int_{-T}^{T} p(t) \cos xt \, dt.$$

By the properties of the masking function we have

$$c(x) = \frac{1}{T}\int_{-\infty}^{\infty} P(t)\, a(t) \cos xt \, dt.$$

Transforming the right-hand side by Parseval's theorem (Wiener 1933, theorem 3, p. 70), we have

$$c(x) = \int_0^{\infty} C(k) \left[\frac{\sin T(x-k)}{T(x-k)} + \frac{\sin T(x+k)}{T(x+k)}\right] dk. \tag{B}$$

Now $c(x)$ is clearly independent of the values of $P(t)$ outside $(-T, T)$, and so the right-hand side of (B) represents a 'weighted mean' common to all the function $C(x)$. Further, a Fourier cosine analysis of $p(t)$ would give the values of $c(s\pi/T)$, ($s = 0, \pm 1, \pm 2, \ldots$), and the other methods of analysis described above would yield other points on the curve $y = c(x)$. We *define* $c(x)$ as the cosine frequency spectrum of the finite record. It is the Fourier cosine transform of the particular continuation of $p(t)$ that is zero outside $(-T, T)$. A sine frequency spectrum $s(x)$ is defined similarly.

Single frequency

Consider the simplest example when $p(t)$ consists of a single sine wave of period π/x_0. One possible continuation is obtained by letting the sine wave run on unbroken to infinity in either direction. For this continuation $C(x)$ is a Dirac delta function near x_0, and zero elsewhere, so that

$$c(x) = \frac{\sin T(x-x_0)}{T(x-x_0)} + \frac{\sin T(x+x_0)}{T(x+x_0)}. \tag{C}$$

If Tx_0 is large, i.e. a large number of wave-lengths are included in the given interval, then the second term on the right of (C) is negligible compared with unity, the amplitude of the sine wave. The first term is a curve of the form $(\sin z)/z$ symmetrical about the ordinate $x = x_0$. The larger Tx_0, the smaller is the 'width' of $c(x)$, i.e. the interval in which $c(x)$ is appreciable. Hence we may say that the effect of the finiteness of the record is to blur every line in the spectrum into a spread pattern $(\sin z)/z$ and the longer the record the narrower is the spread pattern. The effects of two or more sine waves are of course additive.

Fourier analysis

If the only automatic means of analysis at our disposal is a Fourier analyzer, we can still obtain as many points on the curve $c(x)$ as we desire. For suppose the record is continued as far as the points $t = \pm T'$, when $T' > T$ and that in the intervening space the function is taken to be zero. This defines a function $P^*(t)$ over the range $(-T', T')$. Now let a Fourier analysis of $P^*(t)$ be made. This is effectively to evaluate the integrals,

$$\frac{1}{T'}\int_{-T'}^{T'} P^*(t)\cos xt\,dt = \frac{1}{T'}\int_{-T}^{T} p(t)\cos xt\,dt = \frac{T}{T'}c(x),$$

where x is now not a multiple of π/T but of π/T'. Hence using only Fourier analysis we can evaluate the spectra $c(x)$ and $s(x)$ at points as finely spaced as we please.

The question now arises how far it is worth while to determine $c(x)$ and $s(x)$ by the method just described. $c(x)$ is the cosine transform of the particular continuation of $p(t)$ with zero beyond $(-T, T)$. If we are to consider the form of $c(x)$ as typical of the transforms $C(x)$ of continuations of $p(t)$ it will not be worth while to determine $c(x)$ in more detail than is common to all the $C(x)$'s. Now by the definition of the Fourier transform

$$C(x) = \frac{1}{\surd(2\pi)}\int_{-\infty}^{\infty} P(t)\cos xt\,dt.$$

Since $P(t)$ is fixed in the range $(-T, T)$ and arbitrary outside the lower frequencies in $C(x)$ are determined and the higher frequencies vary. Hence we may say that the $C(x)$ differ by functions that oscillate with frequency greater than $2\pi/T$. Alternatively, since $2\pi/T$ is roughly the 'width' of the smoothing function $\dfrac{\sin T(x-x_0)}{T(x-x_0)}$ in (B), we should expect variations in $c(x)$ of less than this frequency to be significant, those of higher frequency not so.

Now the ordinary Fourier analysis of a record gives points on the spectrum at intervals of length π/T, which is just half the period of the smallest 'significant' oscillation. We conclude that if there are any significant trends in the frequency characteristics of a record of finite length, then the Fourier analysis will serve to reveal them.

Some Fourier analyzers give only the amplitude spectrum $r(x)$, where

$$[r(x)]^2 = [c(x)]^2 + [s(x)]^2.$$

We have considered above only the cosine spectrum $c(x)$, but similar arguments would apply to the sine spectrum and to the amplitude spectrum.

Appendix 3. The length and spacing of pressure records

In this note we shall consider the optimum length of a typical pressure record and the optimum interval of time that should elapse between successive records. Both of those depend partly on the particular band of swell that we wish to study.

Consider first the length of the record. From the argument of appendix 2 two sine waves are adequately resolved only if their frequencies differ by at least $2\pi/T$, where T is half the length of the record. Hence in general the longer the record the better is the resolution.

However, the bands of swell from any given storm undergo a slow change in frequency due partly to the Cauchy-Poisson shortening of period and partly to the changing tidal streams near the coast. If the frequency shifts too far during the length of one record then each band of swell will become blurred and resolution will again be lost. This consideration, therefore, sets an upper limit to the length of the record. We shall now investigate the extent of the blurring by finding the frequency spectrum of a slowly accelerated sine wave.

Let
$$p(t) = \sin(kt + ft^2 + \phi) \quad (-T < t < T)$$

be a constantly accelerated sine wave. The wave of constant frequency and of the same phase and wave-length as $p(t)$ at the middle of the record is given by

$$q(t) = \sin(kt + \phi).$$

The phase difference between these two at either end of the record is fT^2. We may assume that $fT^2 < \pi$, as it will appear that for much smaller phase differences the blurring effect is serious.

The frequency analysis of $p(t)$ (see appendix 2) is given by the functions

$$A(m) = \frac{1}{T}\int_{-T}^{T} \sin(kt + ft^2 + \phi)\cos mt\,dt$$

and
$$B(m) = \frac{1}{T}\int_{-T}^{T} \sin(kt + ft^2 + \phi)\sin mt\,dt.$$

We shall write

$$A(m) = \frac{1}{2T}\int_{-T}^{T} \sin[(k-m)\,t + ft^2 + \phi]\,dt + \frac{1}{2T}\int_{-T}^{T} \sin[(k+m)\,t + ft^2 + \phi]\,dt$$
$$= \qquad A_1(m) \qquad + \qquad A_2(m),$$

$$B(m) = \frac{1}{2T}\int_{-T}^{T} \cos[(k-m)\,t + ft^2 + \phi]\,dt - \frac{1}{2T}\int_{-T}^{T} \cos[(k+m)\,t + ft^2 + \phi]\,dt$$
$$= \qquad B_1(m) \qquad - \qquad B_2(m).$$

Now $A_2(m)$ and $B_2(m)$ are both small compared with unity, the amplitude of the regular wave. For

$$\{A_2(m)\}^2+\{B_2(m)\}^2 = \frac{1}{T^2}\left[\int_0^T \cos(k+m)\,t\cos ft^2\,dt\right]^2+\frac{1}{T^2}\left[\int_0^T \cos(k+m)\,t\sin ft^2\,dt\right]^2.$$

Dividing the range of integration into parts (two at most) for which $\cos ft^2$ and $\sin ft^2$ are monotonic and applying the second mean-value theorem to each part we have

$$\left|\frac{1}{T}\int_0^T \cos(k+m)\,t\cos ft^2\,dt\right| \leqslant \frac{3}{(k+m)\,T}, \quad \left|\frac{1}{T}\int_0^T \cos(k+m)\,t\sin ft^2\,dt\right| \leqslant \frac{4}{(k+m)\,T},$$

and, therefore,

$$[\{A_2(m)\}^2+\{B_2(m)\}^2]^{\frac{1}{2}} \leqslant \frac{5}{(k+m)\,T}.$$

If r is the order of the harmonic considered

$$kT = \pi r.$$

Thus, assuming m positive,

$$\frac{5}{(k+m)\,T} \leqslant \frac{5}{\pi r},$$

and for the 60th harmonic

$$[\{A_2(m)\}^2+\{B_2(m)\}^2]^{\frac{1}{2}} \leqslant 0{\cdot}03,$$

which is negligible. The amplitude spectrum $R(m)$ is, therefore, given by

$$\{R(m)\}^2 = \{A(m)\}^2+\{B(m)\}^2$$
$$= \{A_1(m)\}^2+\{B_1(m)\}^2.$$

$R(m)$ is clearly independent of ϕ so that we may put ϕ equal to zero. Then writing

$$C(u) = \int_0^u \cos\frac{\pi}{2}v^2\,dv, \quad S(u) = \int_0^u \sin\frac{\pi}{2}v^2\,dv$$

and

$$a = T\sqrt{\left(\frac{2f}{\pi}\right)}, \quad x = \frac{k-m}{\sqrt{(2\pi f)}},$$

the expression $\{A_1(m)\}^2+\{B_1(m)\}^2$ can be reduced to

$$\frac{[C(x+a)-C(x-a)]^2+[S(x+a)-S(x-a)]^2}{4a^2}.$$

This is the spread pattern for the accelerated wave. When there is no acceleration ($f=0$) the expression reduces to the usual form

$$\{R(\infty, 0)\}^2 = \frac{\sin^2\pi X}{\pi^2 X^2},$$

where

$$X = ax = \frac{T(k-m)}{\pi}$$

has a finite value. This is the ordinary spread pattern for a regular wave (see appendix 2).

The shape of the spread pattern depends only on the parameter a, or $T\sqrt{(2f/\pi)}$, that is on the phase change $\pi a^2/2 = fT^2$ between the middle and either end of the record. Figure 16 shows the spread pattern for several values of a. As might be expected, the spread pattern lies mainly between the limits $\pm a$, which indicate the frequencies of the accelerated wave at the beginning and end of the record. As the length of the record increases, the amplitude of the central peak falls sharply while that of the flanking peaks is augmented, reducing greatly the resolving power of the analysis. It is clear from the curves that the loss of resolving power is serious even when a is as small as 1·4. The maximum allowable value of a is probably about 1·0, corresponding to a phase shift of $\frac{1}{2}\pi$ between the middle and end of the record.

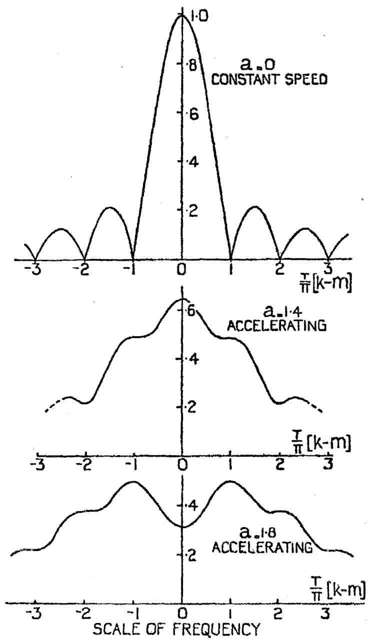

FIGURE 16. The spread pattern in the spectrum of an accelerated wave.

We can now determine the maximum duration of a typical record. The waves from a Cauchy-Poisson disturbance at distance R are of the form $\cos gt^2/4R$. The phase shift, as defined above, between the middle and end of the record is then $gT^2/4R$. If we take a storm distant 1000 miles the phase shift from this cause is $20T^2$, where T is measured in hours. Also the maximum tidal stream velocity is about $1\frac{1}{2}$ knots or $2\frac{1}{2}$ ft./sec., and since the tidal period is $12\frac{1}{2}$ hr. the maximum rate of change of velocity is about $2\pi \times 2\frac{1}{2}/12\frac{1}{2}$ ft./sec./hr. or 4500 ft./hr.^{2}. Since the stream may be changing in velocity at the point where the wave-train enters it, as well as at the recording station, the effect is equivalent to an acceleration of the observer which may be of twice this amount, or 9000 ft./hr.^{2}. With this acceleration the observer would in T hours have moved through a distance of $4500T^2$ ft. Hence, for a typical short wave of length say 500 ft. the maximum phase shift due to the tidal streams will be

$$4500 \times T^2 \times \frac{2\pi}{500} = 57T^2.$$

Since the total phase shift is to be less than $\frac{1}{2}\pi$ we have altogether

$$77T^2 \leqslant \tfrac{1}{2}\pi,$$

giving the maximum length $2T$ of the record as 17·1 min. This would be a suitable length of record for the study of short swell from a fairly close storm. If we only wished to study the longer swell from a more distant storm it would be permissible to increase the length of the record and so to improve the resolution. The advisability of choosing such a low value as 17 min. for the length of the record is mainly due to the effect on the wave period of the changing tidal streams; but the tidal stream does not always exert its full effect, as for example at neaps or when the waves approach the coast at an angle to the streams. If the effect of tidal streams is neglected the optimum duration is 33 min. for a storm 1000 miles away or 47 min. for a storm 2000 miles away.

We shall consider now the spacing of the records. It will not be necessary for ordinary purposes to take records more frequently than will show a significant shift of any given frequency band from one record to the next. Also if we wish to study the swell closely, records should ideally be taken at least as often as this. We shall say that a frequency band has shifted significantly when it has moved through at least two harmonics of the record, that is a frequency interval of $1/T$.

With these criteria for the adequate resolution of the frequencies it can be shown that, ideally, successive records should follow continuously upon one another, so that we are analyzing successive strips of a continuous record. Only the length of the strips depends on the swell to be studied. We may assume that over a short length of time the acceleration of the record is constant, so that any given frequency band is of the form $\sin(kt+ft^2+\phi)$. The frequency change in time t is then ft/π so that the time T_1 that should elapse between the centres of successive records is given by

$$\frac{fT_1}{\pi} = \frac{1}{T}, \tag{D}$$

$2T$ being the length of a typical record. But if T is determined by our former considerations the phase change between middle and end of a record is given by

$$fT^2 = \tfrac{1}{2}\pi, \quad \text{and so} \quad f = \frac{\pi}{2T^2}.$$

Substituting in (D) we have

$$\frac{\pi}{2T^2}\frac{T_1}{\pi} = \frac{1}{T} \quad \text{or} \quad T_1 = 2T,$$

so that the records follow continuously.

This situation only arises because of the criteria for blurring that we happen to have chosen. If we chose, say, $\frac{1}{4}\pi$ instead of $\frac{1}{2}\pi$ as the maximum allowable phase shift in a record, the spacing T_1 as determined by (D) would be wider. Consequently there would be a gap between successive records. The same would happen if we were content to allow a frequency change of more than $1/T$ between two consecutive records.

During the succession of records taken during May 1946 the length of the record was taken as 20 min., but it was not possible for practical reasons to take records more frequently than once every 2 hr.

Appendix 4. The effect of decreasing soundings on travel time

When waves enter shoaling water their rate of travel is changed; the effect is to increase the velocity of the wave groups so long as the shoaling is not too great. The continental shelf extends westward from Cornwall for a distance of about 200 miles, the mean depth of water being about 60 fathoms, and the wave-trains from a distant storm arrive at the recording station somewhat earlier than if they had travelled in deep water throughout their whole journey. Table 1 shows the calculated time differences, the positive sign implying that the waves arrive earlier, and the negative sign implying that they arrive later than would be the case if the water were very deep. It is only the very long waves of period greater than 25 sec. that are delayed by the continental shelf.

The table has been calculated from the formulae

$$c^2 = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}, \quad c_g = c - \lambda \left(\frac{\partial c}{\partial \lambda}\right),$$

where c = wave velocity, c_g = group velocity, h = water depth, λ = wave-length.

Table 1

wave period (sec.)	group velocity (knots)		time difference (hr.) in travelling 200 miles
	deep water	60 fathoms	
10	15·2	16·5	+0·97
12	18·1	20·9	+1·48
14	21·1	25·5	+1·64
16	24·1	29·2	+1·45
18	27·2	31·7	+1·06
20	30·2	33·9	+0·73
22	33·2	35·7	+0·43
24	36·2	37·2	+0·13
26	39·2	38·5	−0·10
28	42·3	39·4	−0·34

The propagation diagrams in the text are drawn on the assumption that the water is very deep throughout the sea. It would be possible to construct the diagrams making allowance for the change in group velocity near the coast but since the total time taken by the waves to reach the coast is very considerable (in the propagation diagram of figure 15 it is more than 100 hr.) it has not been thought necessary to make a correction amounting to 1 or 2 %.

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